On the basis property of the root function systems of regular boundary value problems for the Sturm-Liouville operator

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Abstract. We consider the nonselfadjoint Sturm-Liouville operator with regular but not strongly regular boundary conditions. We examine the basis property of the root function system of the mentioned operator.

In the present paper we study eigenvalue problems for the nonselfadjoint Sturm-Liouville operator

$$Lu = u'' - q(x)u \tag{1}$$

defined on the interval $(0,1)$, where $q(x)$ is an arbitrary complex-valued function of the class $L_1(0,1)$. Our main purpose is to investigate the basis property of the root function system of operator (1) with regular but not strongly regular boundary conditions. Author's interest to this problem was stimulated by the papers of V.A. Il'in [1-3].

By $\varphi(x,\mu), \psi(x,\mu)$ we denote the fundamental for $\mu \neq 0$ system of solutions to the equation

$$u'' - q(x)u + \mu^2 u = 0$$

determined by the initial conditions $\varphi(0,\mu) = \psi(0,\mu) = 1$, $\varphi'_x(0,\mu) = i\mu$, $\psi'_x(0,\mu) = -i\mu$. It is well known that the functions $\varphi(x,\mu)$ and $\psi(x,\mu)$ satisfy the integral equations

$$\varphi(x,\mu) = e^{i\mu x} + \frac{1}{\mu}\int_0^x \sin\mu(x-t)q(t)\varphi(t,\mu)dt, \tag{2}$$

$$\psi(x,\mu) = e^{-i\mu x} + \frac{1}{\mu}\int_0^x \sin\mu(x-t)q(t)\psi(t,\mu)dt, \tag{2'}$$

respectively. Also it is well known that these functions are continuous with their partial derivatives, and for any fixed x they are analytic functions of the parameter μ. Later on, we suppose that the inequality

$$|Im\mu| < M \tag{3}$$

holds, where M is some constant. It is known [4] that the estimates

$$|\varphi(x,\mu)| \le c_1, \qquad |\varphi'_\mu(x,\mu)| \le c_1, \tag{4}$$

$$|\psi(x,\mu)| \le c_2, \qquad |\psi'_\mu(x,\mu)| \le c_2 \tag{4'}$$

are valid for $0 \le x \le 1$ and $|\mu| \ge \mu_0$, where μ_0 is a sufficiently large number.

We need more precise asymptotic formulas for the functions $\varphi(x,\mu)$ and $\psi(x,\mu)$. Transforming the right-hand side of (2), we have

$$\begin{gathered}\varphi(x,\mu) = e^{i\mu x} + (2i\mu)^{-1}\int_0^x (e^{i\mu(x-t)} - e^{-i\mu(x-t)})q(t)\times \\ \times[e^{i\mu t} + \mu^{-1}\int_0^t \sin\mu(t-s)q(s)\varphi(t,\mu)ds]dt = \\ = e^{i\mu x}\{1 + (2i\mu)^{-1}\int_0^x q(t)dt - (2i\mu)^{-1}\int_0^x e^{2i\mu(t-x)}q(t)dt - \\ -(4\mu^2)^{-1}\int_0^x (e^{i\mu(x-t)} - e^{-i\mu(x-t)})q(t)dt\times \\ \times\int_0^t (e^{i\mu(t-s)} - e^{-i\mu(t-s)})q(s)[e^{i\mu s} + \mu^{-1}\int_0^s \sin\mu(s-y)q(y)\varphi(y,\mu)dy]ds\}.\end{gathered} \tag{5}$$

We denote the sum of the first three summands in braces on the right-hand side of (5) by $F_1(x,\mu)$. Dividing the integral with respect to s on the right-hand side of (5) into three summands, we obtain

$$\begin{gathered}\varphi(x,\mu) = e^{i\mu x}\{F_1(x,\mu) - (4\mu^2)^{-1}e^{-i\mu x}\int_0^x (e^{i\mu(x-t)} - e^{-i\mu(x-t)})q(t)dt\times \\ \times[e^{i\mu t}\int_0^t q(s)ds - \int_0^t e^{-i\mu(t-2s)}q(s)ds + \\ +\mu^{-1}\int_0^t (e^{i\mu(t-s)} - e^{-i\mu(t-s)})q(s)ds\int_0^s \sin\mu(s-y)q(y)\varphi(y,\mu)dy]\}.\end{gathered} \tag{6}$$

We denote the last summand in square brackets on the right-hand side of (6) by $\theta_1(t,\mu)$. It follows from (3) and (4) that $\theta_1(t,\mu) = O(\mu^{-1})$ and $\partial\theta_1(t,\mu)/\partial\mu = O(\mu^{-1})$.

Writing the terms in braces on the right-hand side of (6) in descending powers of μ, we get

$$\begin{gathered}\varphi(x,\mu)=e^{i\mu x}\{F_1(x,\mu)-(4\mu^2)^{-1}\int_0^x q(t)dt\int_0^t q(s)ds+\\ +(4\mu^2)^{-1}\int_0^x e^{2i\mu(t-x)}q(t)dt\int_0^t q(s)ds+\\ +(4\mu^2)^{-1}e^{-i\mu x}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)dt\int_0^t e^{-i\mu(t-2s)}q(s)ds-\\ -(4\mu^2)^{-1}e^{-i\mu x}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)\theta_1(t,\mu)dt\}.\end{gathered}\tag{7}$$

We denote the last summand in braces on the right-hand side of (7) by $\theta_2(x,\mu)$. It is readily seen that $\theta_2(x,\mu)=O(\mu^{-3})$ and $\partial\theta_2(x,\mu)/\partial\mu=O(\mu^{-3})$. Simplifying the expression in braces on the right-hand side of (7), we obtain

$$\begin{gathered}\varphi(x,\mu)=e^{i\mu x}\{F_1(x,\mu)-(4\mu^2)^{-1}\int_0^x q(t)dt\int_0^t q(s)ds+\\ +[(4\mu^2)^{-1}(e^{-2i\mu x}\int_0^x e^{2i\mu t}q(t)dt\int_0^t q(s)ds+\int_0^x e^{-2i\mu t}q(t)dt\int_0^t e^{2i\mu s}q(s)ds-\\ -e^{-2i\mu x}\int_0^x q(t)dt\int_0^t e^{2i\mu s}q(s)ds)]+\theta_2(x,\mu)\}.\end{gathered}\tag{8}$$

We denote the expression in square brackets on the right-hand side of (8) by $\theta_3(x,\mu)$. It follows from (3) and the Riemann lemma that $\theta_3(x,\mu)=o(\mu^{-2})$ and $\partial\theta_3(x,\mu)/\partial\mu=o(\mu^{-2})$.

Transforming the right-hand side of (2′), we have

$$\begin{gathered}\psi(x,\mu)=e^{-i\mu x}+(2i\mu)^{-1}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)\times\\ \times[e^{-i\mu t}+\mu^{-1}\int_0^t\sin\mu(t-s)q(s)\psi(s,\mu)ds]dt=\\ =e^{-i\mu x}\{1-(2i\mu)^{-1}\int_0^x q(t)dt+(2i\mu)^{-1}\int_0^x e^{2i\mu(x-t)}q(t)dt-\\ -(4\mu^2)^{-1}e^{i\mu x}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)dt\times\\ \times\int_0^t(e^{i\mu(t-s)}-e^{-i\mu(t-s)})q(s)[e^{-i\mu s}+\mu^{-1}\int_0^s\sin\mu(s-y)q(y)\psi(y,\mu)dy]ds\}.\end{gathered}\tag{5′}$$

We denote the sum of the first three summands in braces on the right-hand side of (5′) by $F_2(x,\mu)$. Dividing the integral with respect to s on

the right-hand side of (5′) into the sum of three summands, we obtain

$$\psi(x,\mu)=e^{-i\mu x}\{F_2(x,\mu)-(4\mu^2)^{-1}e^{i\mu x}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)dt\times$$
$$\times[-e^{-i\mu t}\int_0^t q(t)dt+\int_0^t e^{i\mu(t-2s)}q(s)ds+$$
$$+\mu^{-1}\int_0^t(e^{i\mu(t-s)}-e^{-i\mu(t-s)})q(s)ds\int_0^s\sin\mu(s-y)q(y)\psi(y,\mu)dy]\}. \tag{6'}$$

We denote the last summand in square brackets on the right-hand side of (6′) by $\tilde{\theta}_1(t,\mu)$. It follows from (3) and (4′) that $\tilde{\theta}_1(t,\mu)=O(\mu^{-1})$ and $\partial\tilde{\theta}_1(t,\mu)/\partial\mu=O(\mu^{-1})$.

Writing the terms in braces on the right-hand side of (6′) in decsending powers of μ, we get

$$\begin{aligned}\psi(x,\mu)=e^{-i\mu x}\{F_2(x,\mu)-(4\mu^2)^{-1}\int_0^x q(t)dt\int_0^t q(s)ds+\\+(4\mu^2)^{-1}\int_0^x e^{2i\mu(x-t)}q(t)dt\int_0^t q(s)ds-\\-(4\mu^2)^{-1}e^{i\mu x}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)dt\int_0^t e^{i\mu(t-2s)}q(s)ds-\\-(4\mu^2)^{-1}e^{i\mu x}\int_0^x(e^{i\mu(x-t)}-e^{-i\mu(x-t)})q(t)\tilde{\theta}_1(t,\mu)dt\}.\end{aligned} \tag{7'}$$

We denote the last summand in braces on the right-hand side (7′) by $\tilde{\theta}(x,\mu)$. It is readily seen that $\tilde{\theta}_2(x,\mu)=O(\mu^{-3})$ and $\partial\tilde{\theta}_2(x,\mu)/\partial\mu=O(\mu^{-3})$.

Simplifying the expression in braces on the right-hand side of (7′), we obtain

$$\psi(x,\mu)=e^{-i\mu x}\{F_2(x,\mu)-(4\mu^2)^{-1}\int_0^x q(t)dt\int_0^t q(s)ds+$$
$$+[(4\mu^2)^{-1}(e^{2i\mu x}\int_0^x e^{-2i\mu t}dt\int_0^t q(s)ds+\int_0^x e^{2i\mu t}q(t)dt\int_0^t e^{-2i\mu s}q(s)ds-$$
$$-e^{2i\mu x}\int_0^x q(t)dt\int_0^t e^{-2i\mu s}q(s)ds)]+\tilde{\theta}_2(x,\mu)\}. \tag{8'}$$

We denote the expression in square brackets on the right-hand side (8′) by $\tilde{\theta}_3(x,\mu)$. It follows from (3) and the Riemann lemma that $\tilde{\theta}_3(x,\mu)=o(\mu^{-2})$ and $\partial\tilde{\theta}_3(x,\mu)/\partial\mu=o(\mu^{-2})$.

In the same way, we get asymptotic formulas for the functions $\varphi'_x(x,\mu)$ and $\psi'_x(x,\mu)$. Differentiating relations (2) and (2′), we have

$$\varphi'_x(x,\mu)=i\mu e^{i\mu x}+\int_0^x\cos\mu(x-t)q(t)\varphi(t,\mu)dt, \tag{9}$$

$$\psi'_x(x,\mu) = -i\mu e^{-i\mu} + \int_0^x \cos\mu(x-t)q(t)\psi(t,\mu)dt. \qquad (9')$$

Transforming the second summand on the right-hand side of (9) according to (8), we have

$$\begin{aligned}\varphi'_x(x,\mu) = i\mu e^{i\mu x} + \tfrac{1}{2}\textstyle\int_0^x (e^{i\mu(x-t)} + e^{-i\mu(x-t)})q(t)\times \\ \times[e^{i\mu t}(F_1(t,\mu)+\theta_4(t,\mu)]dt = \\ = i\mu e^{i\mu x} + \tfrac{1}{2}\textstyle\int_0^x (e^{i\mu x} + e^{i\mu(2t-x)})q(t)[F_1(t,\mu)+\theta_4(t,\mu)]dt,\end{aligned} \qquad (10)$$

where

$$\theta_4(t,\mu) = -(4\mu^2)^{-1}\int_0^t q(s)ds\int_0^s q(y)dy + \theta_2(t,\mu) + \theta_3(t,\mu).$$

Evidently, $\theta_4(t,\mu) = O(\mu^{-2})$ and $\partial\theta_4(t,\mu)/\partial\mu = O(\mu^{-2})$.

Writing the terms on the right-hand side of (10) in decsending powers of μ, we get

$$\begin{aligned}\varphi'_x(x,\mu) = e^{i\mu x}\{i\mu + \tfrac{1}{2}\textstyle\int_0^x q(t)dt + \tfrac{1}{2}\int_0^x e^{2i\mu(t-x)}q(t)dt + \\ +(4i\mu)^{-1}\textstyle\int_0^x q(t)dt\int_0^t q(s)ds + \\ +[-(4i\mu)^{-1}\textstyle\int_0^x e^{-2i\mu t}q(t)dt\int_0^t e^{2i\mu s}q(s)ds + \\ +(4i\mu)^{-1}e^{-2i\mu x}\textstyle\int_0^x e^{2i\mu t}q(t)dt\int_0^t q(s)ds - \\ -(4i\mu)^{-1}e^{-2i\mu x}\textstyle\int_0^x q(t)dt\int_0^t e^{2i\mu s}q(s)ds] + \\ +\tfrac{1}{2}\textstyle\int_0^x (1+e^{2i\mu(t-x)})q(t)\theta_4(t,\mu)dt\}.\end{aligned} \qquad (11)$$

We denote the expression in square brackets of (11) by $\theta_5(x,\mu)$. It follows from (3) and the Riemann lemma that $\theta_5(x,\mu) = o(\mu^{-1})$ and $\partial\theta_5(x,\mu)/\partial\mu = o(\mu^{-1})$.

Trasforming the second summand on the right-hand side of (9′) according to (8′), we obtain

$$\begin{aligned}\psi'_x(x,\mu) = -i\mu e^{-i\mu x} + \tfrac{1}{2}\textstyle\int_0^x (e^{i\mu(x-t)} + e^{-i\mu(x-t)})q(t)\times \\ \times[e^{-i\mu t}(F_2(t,\mu)+\tilde{\theta}(t,\mu)]dt = \\ = -i\mu e^{-i\mu x} + \tfrac{1}{2}\textstyle\int_0^x (e^{i\mu(x-2t)} + e^{-i\mu x})q(t)[F_2(t,\mu)+\tilde{\theta}_4(t,\mu)]dt,\end{aligned} \qquad (10')$$

where

$$\tilde{\theta}_4(t,\mu) = -(4\mu^2)^{-1}\int_0^t q(s)ds\int_0^s q(y)dy + \tilde{\theta}_2(t,\mu) + \tilde{\theta}_3(t,\mu).$$

Obviously,

$$\tilde{\theta}_4(t,\mu) = O(\mu^{-2}), \qquad \partial\tilde{\theta}_4(t,\mu)/\partial\mu = O(\mu^{-2}).$$

Writing the terms on the right-hand side of $(10')$ in decsending powers of μ, we get

$$\begin{aligned}\psi'_x(x,\mu) = e^{-i\mu x}\{-i\mu + \tfrac{1}{2}\textstyle\int_0^x q(t)dt + \tfrac{1}{2}\int_0^x e^{2i\mu(x-t)}q(t)dt- \\ -(4i\mu)^{-1}\textstyle\int_0^x q(t)dt\int_0^t q(s)ds+ \\ +[(4i\mu)^{-1}\textstyle\int_0^x e^{2i\mu t}q(t)dt\int_0^t e^{-2i\mu s}q(s)ds- \\ -(4i\mu)^{-1}e^{2i\mu x}\textstyle\int_0^x e^{-2i\mu t}q(t)dt\int_0^t q(s)ds+ \\ +(4i\mu)^{-1}e^{2i\mu x}\textstyle\int_0^x q(t)dt\int_0^t e^{-2i\mu s}q(s)ds]+ \\ +\tfrac{1}{2}\textstyle\int_0^x(e^{2i\mu(x-t)}+1)q(t)\tilde{\theta}_4(t,\mu)\}.\end{aligned} \tag{11'}$$

We denote the expression in square brackets in $(11')$ by $\tilde{\theta}_5(x,\mu)$. It folows from (3) and the Riemann lemma that

$$\tilde{\theta}_5(x,\mu) = o(\mu^{-1}), \qquad \partial\tilde{\theta}_5(x,\mu)/\partial\mu = o(\mu^{-1}).$$

To simplify the asymptotic formulas obtained above, let us prove that

$$\int_0^x q(t)dt\int_0^t q(s)ds = \frac{1}{2}\left(\int_0^x q(t)dt\right)^2. \tag{12}$$

If the function $q(t)$ is continuous on the segment $[0,1]$, then equality (12) can be obtained by integration by parts. In general case, let us approximate the function $q(t)$ by a continuous function $f(t)$ so that $q(t)-f(t)=r(t)$, where $\int_0^1|r(t)|dt<\varepsilon$, where $\varepsilon>0$ is an arbitrary

preassigned number. Denote $q_0 = \int_0^1 |q(t)|dt$. Then we have

$$
\begin{gathered}
|\int_0^x q(t)dt \int_0^t q(s)ds - \frac{1}{2}\left(\int_0^x q(t)dt\right)^2| = \\
= |\int_0^x (f(t)+r(t))dt \int_0^t (f(s)+r(s))ds - \frac{1}{2}\left(\int_0^x (f(t)+r(t))dt\right)^2| = \\
= |\int_0^x f(t)dt \int_0^t f(s)ds + \int_0^x r(t)dt \int_0^t f(s)ds + \int_0^x f(t)dt \int_0^t r(s)ds + \\
+ \int_0^x r(t)dt \int_0^t r(s)ds - \frac{1}{2}\left(\int_0^x f(t)dt\right)^2 - \int_0^x f(t)dt \int_0^x r(t)dt - \\
- \frac{1}{2}\left(\int_0^x r(t)dt\right)^2 | \le \varepsilon(3q_0 + 3\varepsilon/2).
\end{gathered}
$$

This yields that equality (12) holds for any function $q(t)$ of the class $L_1(0,1)$. It follows from (8), (8′), (11), (11′), the estimates for the functions θ_i, $\tilde{\theta}_i$ $(i = \overline{1,5})$ and (12) that

$$
\begin{gathered}
\varphi(x,\mu) = e^{i\mu x}\{1 + (2i\mu)^{-1}\int_0^x q(t)dt - (2i\mu)^{-1}\int_0^x e^{2i\mu(t-x)}q(t)dt - \\
-(8\mu^2)^{-1}\left(\int_0^x q(t)dt\right)^2 + \theta_6(x,\mu)\},
\end{gathered}
\tag{13}
$$

$$
\begin{gathered}
\psi(x,\mu) = e^{-i\mu x}\{1 - (2i\mu)^{-1}\int_0^x q(t)dt + (2i\mu)^{-1}\int_0^x e^{2i\mu(x-t)}q(t)dt - \\
-(8\mu^2)^{-1}\left(\int_0^x q(t)dt\right)^2 + \tilde{\theta}_6(x,\mu)\},
\end{gathered}
\tag{13′}
$$

where $\theta_6(x,\mu) = o(\mu^{-2})$, $\partial\theta_6(x,\mu)/\partial\mu = o(\mu^{-2})$, $\tilde{\theta}_6(x,\mu) = o(\mu^{-2})$, $\partial\tilde{\theta}_6(x,\mu)/\partial\mu = o(\mu^{-2})$;

$$
\begin{gathered}
\varphi'_x(x,\mu) = e^{i\mu x}\{i\mu + \frac{1}{2}\int_0^x q(t)dt + \frac{1}{2}\int_0^x e^{2i\mu(t-x)}q(t)dt + \\
+(8i\mu)^{-1}\left(\int_0^x q(t)dt\right)^2 + \theta_7(x,\mu)\},
\end{gathered}
\tag{14}
$$

$$
\begin{gathered}
\psi'_x(x,\mu) = e^{-i\mu x}\{-i\mu + \frac{1}{2}\int_0^x q(t)dt + \frac{1}{2}\int_0^x e^{2i\mu(x-t)}q(t)dt - \\
-(8i\mu)^{-1}\left(\int_0^x q(t)dt\right)^2 + \tilde{\theta}_7(x,\mu)\},
\end{gathered}
\tag{14′}
$$

where $\theta_7(x,\mu) = o(\mu^{-1})$, $\partial\theta_7(x,\mu)/\partial\mu = o(\mu^{-1})$, $\tilde{\theta}_7(x,\mu) = o(\mu^{-1})$, $\partial\tilde{\theta}_7(x,\mu)/\partial\mu = o(\mu^{-1})$.

For operator (1) let us consider the following two-point boundary value problem with boundary conditions determined by linearly independent forms with arbitrary complex-valued coefficients

$$
\begin{gathered}
B_1(u) = a_1u'(0) + b_1u'(1) + a_0u(0) + b_0u(1) = 0, \\
B_2(u) = c_1u'(0) + d_1u'(1) + c_0u(0) + d_0u(1) = 0.
\end{gathered}
\tag{15}
$$

It is convenient to rewrite conditions (15) in terms of the matrix A, where

$$A = \begin{pmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{pmatrix};$$

by $A(ij)$ we denote the matrix consisting of the ith and jth columns of the matrix A $(1 \le i \le j \le 4)$, and we set $A_{ij} = det\ A(ij)$. We also denote

$$B_1^\star(u) = -a_1 u'(0) + b_1 u'(1) - a_0 u(0) + b_0 u(1),$$
$$B_2^\star(u) = -c_1 u'(0) + d_1 u'(1) - c_0 u(0) + d_0 u(0).$$

In addition, we assume that

$$\int_0^1 q(x)dx = 0. \tag{16}$$

Then it follows from (13), (13′), (14), (14′) and (16) that

$$\begin{array}{ll} \varphi(0,\mu) = 1, & \varphi(1,\mu) = e^{i\mu}(1+P), \\ \varphi'_x(0,\mu) = i\mu, & \varphi'_x(1,\mu) = e^{i\mu}(i\mu + P'), \end{array} \tag{17}$$

where

$$\begin{array}{c} P = -(2i\mu)^{-1} e^{-2i\mu} \int_0^1 e^{2i\mu t} q(t)dt + \theta_6(1,\mu), \\ P' = \frac{1}{2} e^{-2i\mu} \int_0^1 e^{2i\mu t} q(t)dt + \theta_7(1,\mu); \end{array} \tag{18}$$

$$\begin{array}{ll} \psi(0,\mu) = 1, & \psi(1,\mu) = e^{-i\mu}(1+Q), \\ \psi'_x(0,\mu) = -i\mu, & \psi'_x(1,\mu) = e^{-i\mu}(-i\mu + Q'), \end{array} \tag{17′}$$

where

$$\begin{array}{c} Q = (2i\mu)^{-1} e^{2i\mu} \int_0^1 e^{-2i\mu t} q(t)dt + \tilde\theta_6(1,\mu), \\ Q' = \frac{1}{2} e^{2i\mu} \int_0^1 e^{-2i\mu t} q(t)dt + \tilde\theta_7(1,\mu). \end{array} \tag{18′}$$

By $\Delta(\mu)$ we denote the characteristic determinant of the problem

$$Lu + \mu^2 u = 0, \quad B_1(u) = 0, \quad B_2(u) = 0, \tag{19}$$

and by $\Delta_0(\mu)$ we denote the characteristic determinant of the problem

$$u'' + \mu^2 u = 0, \quad B_1(u) = 0, \quad B_2(u) = 0.$$

Using relations (17-18′), and performing some simple though awkward manipulations, we obtain

$$\Delta(\mu) = B_1(\varphi)B_2(\psi) - B_1(\psi)B_2(\varphi) =$$
$$= [a_1 i\mu + b_1 e^{i\mu}(i\mu + P') + a_0 + b_0 e^{i\mu}(1 + P)]\times$$
$$\times[c_1(-i\mu) + d_1 e^{-i\mu}(-i\mu + Q') + c_0 + d_0 e^{-i\mu}(1 + Q)]-$$
$$-[a_1(-i\mu) + b_1 e^{-i\mu}(-i\mu)(-i\mu + Q') + a_0 + b_0 e^{-i\mu}(1 + Q)]\times$$
$$\times[c_1 i\mu + d_1 e^{i\mu}(i\mu + P') + c_0 + d_0 e^{i\mu}(1 + P)] =$$
$$= \Delta_0(\mu) + i\mu A_{12}(e^{i\mu}P' + e^{-i\mu}Q') + i\mu A^{14}(e^{i\mu}P + e^{-i\mu}Q)+$$
$$+A_{23}(e^{i\mu}P' - e^{-i\mu}q') + A_{24}[P' - Q' + i\mu(P + Q)] + \alpha_0(\mu),$$

where

$$\alpha_0(\mu) = b_1 d_1 P'Q' + b_1 d_0 P'Q + b_0 d_1 PQ' + b_0 d_0 PQ-$$
$$-(b_1 d_1 Q'P' + b_1 d_0 Q'P + b_0 d_1 QP' + b_0 d_0 QP) =$$
$$= b_1 d_0(P'Q - PQ') + b_0 d_1(PQ' - P'Q) = A_{24}(P'Q - PQ').$$

It follows from (18) and (18′) that $\alpha_0(\mu) = o(\mu^{-1})$ and $\alpha_0'(\mu) = o(\mu^{-1})$. In the following, we assume that $A_{12} = 0$. It follows from (18) and (18′) that

$$i\mu A_{14}(e^{i\mu}P + e^{-i\mu}Q) =$$
$$= \tfrac{1}{2}A_{14}(e^{i\mu}\int_0^1 e^{-2i\mu t}q(t)dt - e^{-i\mu}\int_0^1 e^{2i\mu t}q(t)dt) + \alpha_1(\mu),$$
$$A_{23}(e^{i\mu}P' - e^{-i\mu}Q') =$$
$$= \tfrac{1}{2}A_{23}(e^{-i\mu}\int_0^1 e^{2i\mu t}q(t)dt - e^{i\mu}\int_0^1 e^{-2i\mu t}q(t)dt) + \alpha_2(\mu),$$
$$A_{24}(P' - Q' + i\mu(P + Q)) = \alpha_3(\mu),$$

where $\alpha_j(\mu) = o(\mu^{-1})$ and $\alpha_j'(\mu) = o(\mu^{-1})$, $j = 1, 2, 3$. Hence,

$$\Delta(\mu) = \Delta_0(\mu) + \frac{1}{2}(A_{14} - A_{23})[e^{i\mu}\int_0^1 e^{-2i\mu t}q(t)dt - e^{-i\mu}\int_0^1 e^{2i\mu t}q(t)dt] + \theta(\mu), \tag{20}$$

where $\theta(\mu) = o(\mu^{-1})$ and $\theta'(\mu) = o(\mu^{-1})$.

Let boundary conditions (15) be regular but not strongly regular [5, pp. 71-73], which, by [5, p. 73] is equivalent to the conditions

$$A_{12} = 0, \quad A_{14} + A_{23} \neq 0, \quad A_{14} + A_{23} = \mp(A_{13} + A_{24}). \tag{21}$$

Without loss of generality, we assume that conditions (15) are normalized [5, p. 66]. This, together with the relation $A_{12} = 0$, yields

$$c_1 = d_1 = 0. \tag{22}$$

Let $\{u_n(x)\}$ be the system of eigenfunctions and associated functions of problem (19), and let $\lambda_n = \mu_n^2$ be the corresponding eigenvalues ($Re\mu_n \geq 0$). By [5, p. 74] the set of numbers μ_n, except for possibly finitely many numbers, consists of two series

$$\mu'_n = 2\pi n + \delta'_n, \quad \mu''_n = 2\pi n + \delta'_n, \tag{23}$$

where $|\delta'_n| \leq c_1 n^{-1/2}$, $|\delta''_n| \leq c_1 n^{-1/2}$, $c_1 > 0$, $n = n_0, n_0 + 1, \ldots$, if $A_{14} + A_{23} = -(A_{13} + A_{24})$ in (21) (case 1), and this set, except for possibly finitely many numbers, consists of two series

$$\tilde{\mu}'_n = (2n-1)\pi + \tilde{\delta}'_n, \quad \tilde{\mu}''_n = (2n-1)\pi + \tilde{\delta}''_n, \tag{23'}$$

where $|\tilde{\delta}'_n| \leq c_3 n^{-1/2}$, $|\tilde{\delta}''_n| \leq c_3 n^{-1/2}$, $c_3 > 0$, $n = n_0, n_0 + 1, \ldots$, if $A_{14} + A_{23} = A_{13} + A_{24}$ in (21) (case 2). It is also known [5, p. 98, p. 91] that the system $\{u_n(x)\}$ is complete in $L_2(0,1)$ and there exists a biorthogonally conjugate system $\{v_n(x)\}$. If μ_n is a simple zero of the function $\Delta(\mu)$, then, by [5, p. 48],

$$u_n(x)\overline{v_n(\xi)} = -2\mu_n H(x, \xi, \mu_n)/\Delta'(\mu_n), \tag{24}$$

where

$$H(x, \xi, \mu) = \begin{vmatrix} \varphi(x) & \psi(x) & g(x, \xi) \\ B_1(\varphi) & B_1(\psi) & B_1(g) \\ B_2(\varphi) & B_2(\psi) & B_2(g) \end{vmatrix}, \tag{25}$$

$$g(x, \xi) = \pm \frac{1}{2W(\xi)} \begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi(\xi) & \psi(\xi) \end{vmatrix}, \quad W(\xi) = \begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(\xi) & \psi(\xi) \end{vmatrix}; \tag{26}$$

moreover, the sign "+" corresponds to the case $x > \xi$, and the sign "-" corresponds to the case $x < \xi$. Developing determinants (25) and (26), we obtain

$$H(x, \xi, \mu_n) = \Phi(x, \xi, \mu_n)/(2W(\xi)), \tag{27}$$

where

$$\Phi(x,\xi,\mu)=\varphi(x)[B_1(\psi)(\psi(\xi)B_2^\star(\varphi)-\varphi(\xi)B_2^\star(\psi))-$$
$$-B_2(\psi)(\psi(\xi)B_1^\star(\varphi)-\varphi(\xi)B_1^\star(\psi))]-\psi(x)[B_1(\varphi)(\psi(\xi)B_2^\star(\varphi)-$$
$$-\varphi(\xi)B_2^\star(\psi))-B_2(\varphi)(\psi(\xi)B_1^\star(\varphi)-\varphi(\xi)B_1^\star(\psi))]. \tag{28}$$

Theorem 1. *If $A_{14}=A_{23}$ and $A_{34}\neq 0$, then the system $\{u_n(x)\}$ forms a Riesz basis in $L_2(0,1)$.*

Proof. It follows from [6] that

$$\Delta_0(\mu)=\mp i(A_{13}+A_{24})\mu e^{-i\mu}(e^{i\mu}\mp 1)[(e^{i\mu}\mp 1)\pm\frac{A_{34}}{i(A_{13}+A_{24})\mu}(e^{i\mu}\pm 1)],$$

where the upper sign is chosen in the case 1 and the lower sign is chosen in the case 2. This, together with (20), reduces the equation $\Delta(\mu)=0$ to the form

$$\mp i(A_{13}+A_{24})\mu(1\mp e^{-i\mu})[(e^{i\mu}\mp 1)\pm\frac{A_{34}}{i(A_{13}+A_{24})\mu}(e^{i\mu}\pm 1)]+\theta(\mu)=0.$$

The last equation is reduced to the form

$$w_1(\mu)=(1-e^{-i\mu})[(e^{i\mu}-1)+\frac{b}{i\mu}(e^{i\mu}+1)]+R_1(\mu)=0 \tag{29}$$

in the case 1, and

$$w_2(\mu)=(1+e^{-i\mu})[(e^{i\mu}+1)-\frac{b}{i\mu}(e^{i\mu}-1)]+R_2(\mu)=0 \tag{29$'$}$$

in the case 2, where $b=A_{34}/(A_{13}+A_{24})$, $R_j(\mu)=o(\mu^{-2})$, $R_j'(\mu)=o(\mu^{-2})$, $j=1,2$.

Let us consider case 1. Substituting $\mu=2\pi n+z$ into (29) and using (23) we find that the function $F_n(z)=g(z)f_n(z)+R_1(2\pi n+z)$, where $g(z)=1-e^{-iz}$, $f_n(z)=e^{iz}-1+b(e^{iz}+1)/(i(2\pi n+z))$, has two roots δ_n' and δ_n'' in the disk $|z|\leq c_1 n^{-1/2}$. Evidently, the function $g(z)$ has a unique root $z=0$ in the same disk, moreover, it follows from [6] that the function $f_n(z)$ has a unique root z_n'' in the same disk,

and $z''_n = O(n^{-1})$. It follows from the last equality and the Maclaurin formula for the function e^{iz} that $z''_n = b/(\pi n) + O(n^{-2})$.

By Γ'_n and Γ''_n we denote the circles of radius $r_n = |b|/(4\pi n)$ centered at 0 and $b/(\pi n)$, respectively. It follows from the Maclaurin formula that for all sufficiently large n for $z \in \Gamma'_n \cup \Gamma''_n$ $|g(z)f_n(z)| \geq c_2 n^{-2}$ ($c_2 > 0$). Therefore, for all sufficiently large n for $z \in \Gamma'_n \cup \Gamma''_n$ $|g(z)f_n(z)| > |F_n(z) - g(z)f_n(z)|$. By the Rouche' theorem, it follows from the last inequality that the functions $g(z)f_n(z)$ and $F_n(z)$ have the same number of zeros in the disks bounded by Γ'_n and Γ''_n, hence, for all sufficiently large n the equation $F_n(z) = 0$ has exactly one root in each disk bounded by Γ'_n or Γ''_n. Thus, we have

$$|\delta'_n| < r_n, \quad |\delta''_n - b/(\pi n)| < r_n, \quad |\mu'_n - \mu''_n| > 2r_n. \tag{30}$$

In case 2 equation (29′) can be analysed in a similar way. Arguing as above, we see that

$$|\tilde{\delta}'_n| < r_n, \quad |\tilde{\delta}''_n + b/(\pi n)| < r_n, \quad |\tilde{\mu}'_n - \tilde{\mu}''_n| > 2r_n \tag{30′}$$

for all sufficiently large n. In particular, it follows from (30) and (30′) that the eigenvalues λ_n are asymptotically simple.

Let us prove that for all sufficiently large n

$$c_4 \leq |\Delta'(\mu_n)| \leq c_5, \tag{31}$$

where $c_4 > 0$ and $c_5 > 0$; here μ_n is an arbitrary root of the equation $\Delta(\mu) = 0$. In case 1 we have $\Delta(\mu) = \beta_1 \mu w_1(\mu)$, where $\beta_1 = -i(A_{13} + A_{24})$, therefore,

$$\Delta'(\mu_n) = \beta_1 \mu_n w'_1(\mu_n). \tag{32}$$

Let us estimate the function $w'_1(\mu)$. If $\mu = 2\pi n + z$, then $w_1(\mu) = F_n(z)$. It follows from (30) and the Maclaurin formula that for $z = \delta'_n$ we have $c_6 \leq |g'(z)| \leq c_7$, $c_8/n \leq |f_n(z)| \leq c_9/n$, $|g(z)| \leq c_{10}/n$, $|f_n(z)| \leq c_{11}/n$, and for $z = \delta''_n$ we have $c_{12} \leq |f_n(z)| \leq c_{13}$, $c_{14}/n \leq |g(z)| \leq c_{15}/n$, $|f_n(z)| \leq c_{16}/n$, $|g'(z)| \leq c_{17}/n$ ($c_j > 0$, $j = \overline{6, 17}$).

This implies that $c_{18}/n \leq |(g(z)(f_n(z))'| \leq c_{19}/n$ if $z = \delta_n'$ or $z = \delta_n''$. It follows from the last inequality and (29) that for the same z we have $c_{20}/n \leq |F_n'(z)| \leq c_{21}/n$ ($c_j > 0$, $j = \overline{18, 21}$). This, together with (32), yields estimate (31). Case 2 can be analyzed in a similar way.

Let us estimate the product $u_n(x)\overline{v_n(\xi)}$. Let $H_0(x, \xi, \mu)$, $g_0(x, \xi)$, $W_0(\xi)$ and $_0(x, \xi, \mu)$ be the functions given by (25-28) with $\varphi(x, \mu)$ and $\psi(x, \mu)$ replaced by $e^{i\mu x}$ and $e^{-i\mu x}$.

Let us prove that

$$H(x, \xi, \mu_n) - H_0(x, \xi, \mu_n) = O(n^{-1}) \tag{33}$$

in case 1. Since $\mu_n = 2\pi n + O(n^{-1})$, it follows from (22) that $\varphi(x, \mu_n) = e^{2\pi i n x} + O(n^{-1})$, $B_1(\varphi(x, \mu_n)) = B_1(e^{2\pi i n x}) + O(1)$, $B_2(\varphi(x, \mu_n)) = B_2(e^{2\pi i n x}) + O(n^{-1})$, $B_1^\star(\varphi(x, \mu_n)) = B_1^\star(e^{2\pi i n x}) + O(1)$, $B_2^\star(\varphi(x, \mu_n)) = B_2^\star(e^{2\pi i n x}) + O(n^{-1})$. Similar estimates are valid for the functions $\psi(x, \mu_n)$ and $e^{-2\pi i n x}$. This, together with (28), yields $\Phi(x, \xi, \mu_n) = \Phi_0(x, \xi, 2\pi n) + O(1)$. It also follows from (22) that $\Phi_0(x, \xi, 2\pi n) = O(n)$. It can easily be checked that $W_0(\xi, 2\pi n) = 4\pi i n$ and $W(\xi, \mu_n) = W_0(\xi, 2\pi n) + O(1)$. The last four relations and formula (27) mean that estimate (33) holds.

From (24) and (33) we obtain

$$u_n(x)\overline{v_n(\xi)} = (-4\pi n H_0(x, \xi, 2\pi n) + O(1))/\Delta'(\mu_n).$$

The expression for $H_0(x, \xi, 2\pi n)$ was computed in [7, c. 329]:

$$-2\pi i n H_0(x, \xi, 2\pi n) = A_{34}(\cos 2\pi n(x - \xi) - \cos 2\pi n(x + \xi))$$

($x \neq \xi$). It follows from the last two relations and (31) that

$$|u_n(x)\overline{v_n(\xi)}| \leq C. \tag{34}$$

By the same argument, we obtain estimate (34) in case 2. It follows from (34) [8] that the system $\{u_n(x)\}$ forms a Riesz basis in $L_2(0, 1)$. Theorem 1 is proved.

It was shown in [6] that any boundary conditions (15) satisfying the requirements of Theorem 1 are equivalent to the boundary conditions specified by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & b_0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad or \quad A = \begin{pmatrix} 1 & 1 & 0 & b_0 \\ 0 & 0 & 1 & 1 \end{pmatrix};$$

in both cases, $b_0 \neq 0$.

Theorem 2. *If*

$$A_{14} \neq A_{23}, \tag{35}$$

then the system of root functions $\{u_n(x)\}$ of problems (19) is a Riesz basis in $L_2(0,1)$ if and only if all but finitely many eigenvalues λ_n are multiple (in other words, they are asymptotically multiple).

Proof. Suppose, the eigenvalues λ_n are asymptotically multiple. It is known [9] that the two-dimensional subspaces corresponding to the pairwise close eigenvalues form a basis in $L_2(0,1)$, which is equivalent to an orthogonal basis. Choosing in each of these subspases corresponding to the multiple eigenvalues an orthonormal basis, we obtain [10, p. 414] that the system of root functions of problem (19), which is the union of all orthogonal bases of mentioned subspases, is a Riesz basis in $L_2(0,1)$.

Suppose, the spectrum is not asymptotically multiple. Then there exists a subsequence of numbers such that for any number n from this subsequence $\mu_n' \neq \mu_n''$. Let $\tilde{u}_n(x)$ be the eigenfunction corresponding to an eigenvalue λ_n' from this subsequence, and let $\tilde{v}_n(x)$ be the function in the biorthogonal system corresponding to $\tilde{u}_n(x)$. Let us estimate the product $\tilde{u}_n(x)\overline{\tilde{v}_n(\xi)}$.

We consider the determinant $\Delta_0(\mu)$. It follows from [6] and (21) that

$$\Delta_0(\mu) = -2i(A_{14} + A_{23})\mu(1 \mp \cos\mu) + 2iA_{34}\sin\mu,$$

where the upper sign is chosen in case 1, and the lower sign is chosen

in case 2. Differentiating, we obtain

$$\Delta_0'(\mu) = -2i(A_{14} + A_{23})(1 \mp \cos\mu \pm \mu\sin\mu) + 2iA_{34}\cos\mu.$$

It follows from the last equality and asymptotic formulas (23) and (23$'$) that $|\Delta_0'(\mu_n')| \le c_1\sqrt{n}$. This, together with (20), yields

$$|\Delta'(\mu_n')| \le c_2\sqrt{n}. \tag{36}$$

Let $H_0(x,\xi,\mu)$, $g_0(x,\xi)$, $W_0(\xi)$ and ${}_0(x,\xi,\mu)$ be the functions given by (25-28) with $\varphi(x,\mu)$ and $\psi(x,\mu)$ replaced by $e^{i\mu x}$ and $e^{-i\mu x}$, respectively.

Let us prove that

$$H(x,\xi,\mu_n') - H_0(x,\xi,2\pi n) = O(n^{-1/2}). \tag{37}$$

in case 1. Since $\mu_n' = 2\pi n + O(n^{-1/2})$, it follows from (22), that $\varphi(x,\mu_n') = e^{2\pi i n x} + O(n^{-1/2})$, $B_1(\varphi(x,\mu_n')) = B_1(e^{2\pi i n x}) + O(n^{1/2})$, $B_2(\varphi(x,\mu_n')) = B_2(e^{2\pi i n x}) + O(n^{-1/2})$, $B_1^\star(\varphi(x,\mu_n')) = B_1^\star(e^{2\pi i n x}) + O(n^{1/2})$, $B_2^\star(\varphi(x,\mu_n')) = B_2^\star(e^{2\pi i n x}) + O(n^{-1/2})$. Similar estimates are valid for the functions $\psi(x,\mu_n')$ and $e^{-2\pi i n x}$. This, together with (28), yields $\Phi(x,\xi,\mu_n') = \Phi_0(x,\xi,2\pi n) + O(n^{1/2})$. It also follows from (22) that $\Phi_0(x,\xi,2\pi n) = O(n)$. It can easily be checked that $W_0(\xi,2\pi n) = 4\pi i n$ and $W(\xi,\mu_n) = W_0(\xi,2\pi n) + O(n^{1/2})$. The last four relations and formula (30) mean that estimate (37) holds.

From (24) and (37) we obtain

$$\tilde{u}_n(x)\overline{\tilde{v}_n(\xi)} = (-4\pi n H_0(x,\xi,2\pi n) + O(n^{1/2}))/\Delta'(\mu_n').$$

The expression for $H_0(x,\xi,2\pi n)$ was computed in [7, c. 329]:

$$-2\pi i n H_0(x,\xi,2\pi n) = A_{34}(\cos 2\pi n(x-\xi) - \cos 2\pi n(x+\xi)) +$$
$$+2\pi n[(A_{14} + A_{23} + 2A_{24})\sin 2\pi n(x-\xi) - (A_{14} - A_{23})\sin 2\pi n(x+\xi)]$$

for $x < \xi$,

$$-2\pi i n H_0(x,\xi,2\pi n) = A_{34}(\cos 2\pi n(x-\xi) - \cos 2\pi n(x+\xi)) +$$
$$+2\pi n[(A_{14} + A_{23} + 2A_{13})\sin 2\pi n(x-\xi) - (A_{14} - A_{23})\sin 2\pi n(x+\xi)]$$

for $x > \xi$. It follows from the last three equalities, (36) and (35) that

$$||\tilde{u}_n||_{L_2(0,1)}||\tilde{v}_n||_{L_2(0,1)} \geq C\sqrt{n},$$

where $C > 0$, and, hence, the root function system $\{u_n(x)\}$ of problem (19) is not a basis in $L_2(0,1)$. Case 2 can be treated in a similar way.

Thus, we have established that conditions (21) and (35) reduce the question about the basis property for the system of eigenfunctions and associated functions to the asymptotic multiplicity of the spectrum. The presence of this property depends essentially on the particular form of the boundary conditions and the function $q(x)$. In the simplest case of $q(x) \equiv 0$, the problem was solved completely in [6]. Below, we cite some results of [6].

Suppose that the boundary conditions in problem (19) satisfy (21), (35), and the condition $A_{34} = 0$. We refer to such problems as problems of type (*).

They have asymptotically multiple spectrum, and any boundary conditions (15) satisfying the requirements mentioned above are equivalent to the boundary conditions determined by the matrix

$$A = \begin{pmatrix} 1 & b_1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \end{pmatrix},$$

where either $b_1 = \mp 1$, $d_0 \neq 1$, and $d_0 \neq -1$; $d_0 = \mp 1$, $b_1 \neq 1$, and $b_1 \neq -1$;

$$A = \begin{pmatrix} 1 & \mp 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad or \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mp 1 \end{pmatrix}.$$

The sign is always upper in case 1 and lower in case 2.

If conditions (21) and (35) hold but $A_{34} \neq 0$, then the spectrum of problem (19) is asymptotically simple, and any boundary conditions (15) satisfying the above requirements are equivalent to those specified by the matrix

$$A = \begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix},$$

where either $b_1 = \mp 1$, $d_0 \neq 1$, $d_0 \neq -1$, and $b_0 \neq 0$; $d_0 = \mp 1$, $b_1 \neq 1$, $b_1 \neq -1$, and $b_0 \neq 0$; or

$$A = \begin{pmatrix} 1 & \mp 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad where \quad a_0 \neq 0,$$

or

$$A = \begin{pmatrix} 0 & 1 & 0 & b_0 \\ 0 & 0 & 1 & \mp 1 \end{pmatrix}, \quad where \quad b_0 \neq 0.$$

The sign is always upper in case 1 and lower in case 2.

Suppose that conditions (21) and (35) hold. Then, in the author's opinion, of great interest is the problem of finding potentials $q(x) \not\equiv 0$ that ensure an asymptotically multiple spectrum . In this relation, we mention the following results.

In [11, 12], it was established that, under the condition

$$q(x) = q(1-x), \tag{38}$$

where $x \in [0,1]$, the spectrum of each of the problems

$$Lu + \lambda u = 0, \quad u'(0) = u'(1), \quad u(0) = bu(1);$$

$$Lu + \lambda u = 0, \quad u'(0) = bu'(1), \quad u(0) = u(1),$$

where $b \neq -1$, coincides with that of the periodic problem

$$Lu + \lambda u = 0, \quad u'(0) = u'(1), \quad u(0) = u(1), \tag{39}$$

and the spectrum of each of the problems

$$Lu + \lambda u = 0, \quad u'(0) + u'(1) = 0, \quad u(0) + bu(1) = 0;$$

$$Lu + \lambda u = 0, \quad u'(0) + bu'(1) = 0, \quad u(0) + u(1) = 0,$$

where $b \neq -1$, coincides with that of the antiperiodic problem

$$Lu + \lambda u = 0, \quad u'(0) + u'(1) = 0, \quad u(0) + u(1) = 0. \tag{40}$$

Therefore, under condition (38) the spectrum of a problem of type (*) coincides with the spectrum of problem (39) or (40).

Let $q(x) \in L_2(0,1)$ be a real-valued function. We denote the eigenvalues of problem (39) by λ_0, λ_n^-, and λ_n^+, where $n = 2k$ and $k = 1, 2, \ldots$, and the eigenvalues of problem (40) by λ_n^- and λ_n^+, where $n = 2k-1$ and $k = 1, 2, \ldots$; in both cases, the eigenvalues are enumerated in nondecreasing order. Let $\gamma_n = \lambda_n^+ - \lambda_n^-$ $(n = 1, 2, \ldots)$ be the length of the spectral gap. In [13] estimates for γ_n were obtained for problems (39) and (40) with the potential

$$q(x) = -\pi^2(4\alpha t \cos 2\pi x + 2\alpha^2 \cos 4\pi x), \tag{41}$$

where α, t are real numbers, and $\alpha \neq 0$ and $t \neq 0$. In particular, in [13] it was shown that for even n

$$\gamma_n = \frac{8\pi^2\alpha^n}{2^n[(n-2)!!]^2}|\cos(\frac{\pi}{2}t)|[1 + O((\log n)^3/n)],$$

and for odd n

$$\gamma_n = \frac{8\pi^2\alpha^n}{2^n[(n-2)!!]^2}\frac{2}{\pi}|\sin(\frac{\pi}{2}t)|[1 + O((\log n)^3/n)].$$

Since for potential (41) condition (38) holds, we see that for any problem of type (*) the parameter t can be chosen so that its spectrum is asymptotically multiple or asymptotically simple.

If boundary conditions satisfy (21) and (35), then it follows from [4] that under supplementary conditions $q(x) \in W_1^1[0,1]$ and $2A_{34}^2 \neq (A_{13} + A_{24})(A_{14} - A_{23})(q(1) - q(0))$ the spectrum of problem (19) is asymptotically simple and the root function system is not a basis. For a problem of type (*) the last condition is equivalent to the condition $q(1) \neq q(0)$. It is readily seen that for the potential determined by (41) $q(1) = q(0)$ for any α and t, hence, in comparison with [4], Theorem 2 of the present paper widens the class of boundary value problems such that the corresponding root function system is not a basis.

It is known [14] that the spectrums of periodic and antiperiodic problems on the segment $[0,1]$ for the Mathieu operator $lu = u'' -$

$2\pi^2 a \cos 2\pi x$, where a is a real number ($a \neq 0$), are simple. It follows from our reasoning that the eigenfunction system of the Mathieu operator with boundary conditions of type (*) is not a basis. It is clear that this example is not covered by [4].

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